A ray theory for wave propagation in a non-uniform medium

By E. J. KATZ

Mechanics Department, The Johns Hopkins University

(Received 12 October 1962 and in revised form 12 January 1963)

The study of wave propagation by geometrical optics is applied to a consideration of propagation through a non-uniform, statistically homogeneous medium. Following the trajectory of a phase point, two effects are examined; the retardation of the phase point by the variable local wave speed along its trajectory and the further retardation and dispersion resulting from its meandering. Mean and mean-squared displacements are obtained to describe the retardation of the wave front, and the dispersion of the phase point from the incident direction.

The theory has application as a correction to the use of geometrical optics wherever the latter can be employed. In particular, it is shown by an estimate of the magnitude of the pertinent parameters that an application may well be found in the study of the tsunami (a long 'shallow' ocean wave).

1. Introduction

The simplest wave propagates with constant vector velocity and amplitude. In many physical instances we are interested in its behaviour when it passes through a steady non-uniform medium. If the non-uniformity is of sufficiently small magnitude and large length scale that the wave velocity and amplitude vary only slightly over one wavelength of the incident plane wave, then the perturbed wave may be considered to be 'locally plane' within the medium. Under such circumstances, we can transform the problem from a consideration of a perturbed wave equation,

$$\frac{\partial^2\phi}{\partial t^2} - c^2(\mathbf{r})\,\nabla^2\phi = 0, \tag{1.1}$$

to a study of the rays of the almost plane waves defined by the scalar wave function $\phi(\mathbf{r}, t)$. The variable $c(\mathbf{r})$ is the local speed of propagation at position $\mathbf{r}$. The equations which describe the rays are (see Landau & Lifshitz 1959, equation (66.5), for example)

$$d\mathbf{r}/dt = c\mathbf{n} \quad \text{and} \quad d\mathbf{k}/dt = -k\nabla c, \tag{1.2}$$

where $\mathbf{k}$ is the wave vector along a ray, $k = |\mathbf{k}| = \omega/c$, ω is the wave frequency, which is constant on a ray, and $\mathbf{n} = \mathbf{k}/k$ is the unit vector tangent to a ray at a point. We will trace the propagation of a phase point along a ray when the local wave speed is a statistically homogeneous function varying slightly throughout some region in space, in order to locate the mean and mean-square position of the wave front as a function of time.

The analysis has application in a number of fields including optics, acoustics and shallow-water wave theory. It is the last of these which attracted the attention of the writer to the problem. After a general description of the phenomenon has been derived and discussed, we shall enumerate the conditions which, when satisfied, allow an analogy between geometrical optics and shallow-water waves. Some estimates will be made for the propagation of tsunamis (ocean waves whose wavelengths are larger than the depth of the deep ocean).

The problem of sound propagation in a turbulent medium has received wide attention and ray techniques have been employed. The book by Chernov (1960), for instance, describes a model approach to ray theory by regarding the ray propagation as a stochastic Markov process, permitting the use of a form of the Fokker–Plank equation. Other ray treatments (Ellison 1951, and Muchmore & Wheelon 1955, are examples) make extensive use, without formal derivation, of a form of equation (2.11*b*) to obtain such quantities as the mean-square angle of arrival of the ray and the intensity fluctuations. We will be concerned with the mean retardation in arrival time of a phase point, and the dispersion of arrival times about the mean. The present study also formulates the problem more rigorously than has been the custom and may therefore have an additional value in its clarification of the physical assumptions implied by these previous analyses.

Wave propagation in a non-uniform medium has also been viewed in a Eulerian frame of reference as a wave-scattering problem. Batchelor (1957), among others, applies this technique to the wave equation of (1.1) as well as a more general case. The contrasting viewpoints can be summarized as follows. In a ray theory it is the incident wave which draws our complete attention while the Eulerian analysis develops the scattered field generated from the incident wave. In the latter, the information obtained about the incident wave field is indirect, inasmuch as the energy appearing in the scattered field is related back to an attenuation of the incident field. That the Eulerian analysis does not explicitly contain the ray theory to be discussed here can be seen from equation (14) of Batchelor. There, the logarithmic decrement of the attenuation of the incident wave is found to be directly proportional to an integral length scale of the inhomogeneity of the medium. Holding everything else constant, while increasing this length scale, one finds the attenuation rate increasing while the change in the properties of the medium within a fixed incident wavelength becomes more gradual. This is accompanied by a simultaneous decrease in the scattering angle (Katz 1962). Thus we have an increasing attenuation rate but the attenuated energy is going into a scattered field which is becoming more difficult to distinguish from the incident field itself. It is in this region where a ray theory is applicable and affords direct insight to the physical phenomenon.

2. The path of a phase point

Along a ray, $\mathbf{k} = \mathbf{n}\omega/c$. When substituted into (1.2), this yields

$$\left.\begin{aligned} d\mathbf{r}/dt &= c\mathbf{n} \\ \text{and} \qquad d\mathbf{n}/dt &= -\nabla c + \mathbf{n}(\mathbf{n}\,.\,\nabla c). \end{aligned}\right\} \tag{2.1}$$

These combine into a non-linear, second-order, differential equation for the path of a phase point

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{r}}{dt}\left(\frac{d\mathbf{r}}{dt}\cdot\nabla(\ln c^2)\right) - \tfrac{1}{2}\nabla c^2. \tag{2.2}$$

Let us identify the trajectory of a particular phase point, which was at $\mathbf{r}_0$ when $t = 0$, as

$$\mathbf{r} = \mathbf{X}(t, \mathbf{r}_0). \tag{2.3}$$

For a prescribed local wave speed everywhere and a set of initial conditions, the position of that phase point at all subsequent times is decribed by

$$\frac{\partial^2\mathbf{X}}{\partial t^2} = \frac{\partial\mathbf{X}}{\partial t}\left(\frac{\partial\mathbf{X}}{\partial t}\cdot\nabla(\ln c^2)\right) - \tfrac{1}{2}\nabla c^2, \tag{2.4}$$

where terms like ∇c^2 are understood to be of the gradient the scalar field $c^2(\mathbf{r})$ evaluated at $\mathbf{r} = \mathbf{X}(t, \mathbf{r}_0)$.

The important function of the local properties is seen to be the square of the local phase speed. Separating it into a mean and a variable part,

$$c^2(\mathbf{r}) = c_0^2(1+\mu(\mathbf{r})), \tag{2.5}$$

we define the mean-square velocity, $c_0^2 = \overline{c^2}$, where the overbar denotes an ensemble average. $\mu(\mathbf{r})$ is a homogeneous random function, with zero mean and a root-mean square value (μ') small compared to unity. It corresponds to the variable part of the 'index of refraction' of the medium.

When $\mu(\mathbf{r}) = 0$ everywhere, (2.4) yields

$$\partial\mathbf{X}/\partial t = \text{a constant vector}, \tag{2.6}$$

and we have returned to the simple plane wave in the uniform medium. This suggests that a solution may be developed as a power series in the scaling factor μ':

$$\mathbf{X}(t, \mathbf{r}_0) = \mathbf{X}^{(0)}(t, \mathbf{r}_0) + \mathbf{X}^{(1)}(t, \mathbf{r}_0)\,\mu' + \mathbf{X}^{(2)}(t, \mathbf{r}_0)\,\mu'^2 + \ldots, \tag{2.7}$$

where the coefficients of the expansion, after averaging, will be functions of the normalized auto-correlation functions of $\mu(\mathbf{r})$, but not the absolute magnitude of the variation. The zero-order solution is the integral of (2.6). If we initially consider the waves to be travelling in the x-direction and fix attention on a phase point passing through the origin, then, (2.7) can be rewritten as

$$\mathbf{X}(t) = c_0 t\mathbf{i}_x + \mathbf{X}^{(1)}(t) + \mathbf{X}^{(2)}(t) + \ldots, \tag{2.8}$$

where $\mathbf{X}^{(n)}$ has replaced $\mathbf{X}^{(n)}\mu'^n$ to simplify the notation. Because the medium will be considered statistically homogeneous, the initial position is of no consequence and can be omitted from the argument.

The obvious objective is to separate (2.4) into a sequence of equations with like dependence on μ'. Before doing this, there is one further problem to resolve. In (2.4), even the quantities whose statistical properties are assumed known (for example, $\nabla\mu$) are to be evaluated along an unknown trajectory. To relate them to a known trajectory, we expand these terms about their values along $\mathbf{X}^{(0)}(t) = c_0 t\mathbf{i}_x$ (the unperturbed solution)

$$\nabla\mu|_{\mathbf{r}=\mathbf{X}} = \nabla\mu|_{\mathbf{r}=\mathbf{X}^{(0)}} + [(\mathbf{X}-\mathbf{X}^{(0)})\cdot\nabla]\,\nabla\mu|_{\mathbf{r}=\mathbf{X}^{(0)}} + \ldots. \tag{2.9}$$

For small μ', we can write

$$\nabla(\ln c^2)|_{\mathbf{r}=\mathbf{X}} = \nabla(\mu - \tfrac{1}{2}\mu^2 + \ldots)|_{\mathbf{r}=\mathbf{X}}, \tag{2.10}$$

and then expand $\nabla\mu^2|_{\mathbf{r}=\mathbf{X}}$, etc. as in (2.9). Now, equating terms of like power, the first and second powers of μ' yield

$$\frac{\partial^2\mathbf{X}^{(1)}}{\partial t^2} = \frac{\partial\mathbf{X}^{(0)}}{\partial t}\left(\frac{\partial\mathbf{X}^{(0)}}{\partial t}.\nabla\mu\right) - \frac{c_0^2}{2}\nabla\mu, \tag{2.11}$$

$$\frac{\partial^2\mathbf{X}^{(2)}}{\partial t^2} = \frac{\partial\mathbf{X}^{(1)}}{\partial t}\left(\frac{\partial\mathbf{X}^{(0)}}{\partial t}.\nabla\mu\right) + \frac{\partial\mathbf{X}^{(0)}}{\partial t}\left(\frac{\partial\mathbf{X}^{(1)}}{\partial t}.\nabla\mu + \frac{\partial\mathbf{X}^{(0)}}{\partial t}.[(\mathbf{X}^{(1)}.\nabla)\nabla\mu - \tfrac{1}{2}\nabla\mu^2]\right) - \tfrac{1}{2}c_0^2(\mathbf{X}^{(1)}.\nabla)\nabla\mu. \tag{2.12}$$

These equations are exact, giving the first two $\mathbf{X}^{(n)}$ in terms of the coefficients of lower order. Further equations in the sequence are not required in the present analysis, though their influence is considered briefly in § 6. Substituting for $\partial\mathbf{X}^{(0)}/\partial t$, the component equations in Cartesian co-ordinates are

$$\frac{\partial^2 X_x^{(1)}}{\partial t^2} = \tfrac{1}{2}c_0^2\frac{\partial\mu}{\partial x}, \tag{2.11a}$$

$$\frac{\partial^2 X_y^{(1)}}{\partial t^2} = -\tfrac{1}{2}c_0^2\frac{\partial\mu}{\partial y}, \quad \frac{\partial^2 X_z^{(1)}}{\partial t^2} = -\tfrac{1}{2}c_0^2\frac{\partial\mu}{\partial z}, \tag{2.11b}$$

$$\frac{\partial^2 X_x^{(2)}}{\partial t^2} = c_0\left(\frac{\partial\mathbf{X}^{(1)}}{\partial t}.\nabla\mu + \frac{\partial X_x^{(1)}}{\partial t}\frac{\partial\mu}{\partial x}\right) + \frac{c_0^2}{2}\left[(\mathbf{X}^{(1)}.\nabla)\frac{\partial\mu}{\partial x} - \frac{\partial\mu^2}{\partial x}\right], \tag{2.12a}$$

$$\frac{\partial^2 X_y^{(2)}}{\partial t^2} = c_0\frac{\partial X_y^{(1)}}{\partial t}\frac{\partial\mu}{\partial x} - \frac{c_0^2}{2}(\mathbf{X}^{(1)}.\nabla)\frac{\partial\mu}{\partial y}, \quad \frac{\partial^2 X_z^{(2)}}{\partial t^2} = c_0\frac{\partial X_z^{(1)}}{\partial t}\frac{\partial\mu}{\partial x} - \frac{c_0^2}{2}(\mathbf{X}^{(1)}.\nabla)\frac{\partial\mu}{\partial z}, \tag{2.12b}$$

where the spatial derivatives are all to be evaluated along the known trajectory.

The separation between the position of the phase point and the unperturbed trajectory, $\mathbf{\Delta}(t)$, is given by

$$\mathbf{\Delta}(t) \equiv \mathbf{X} - \mathbf{X}^{(0)} = \mathbf{X}^{(1)} + \mathbf{X}^{(2)} + \ldots, \tag{2.13}$$

and we shall proceed to obtain $\overline{\mathbf{\Delta}(t)}$ and $\overline{\mathbf{\Delta}^2(t)}$ to their first, non-trivial, order of approximation.

3. Wave retardation by a one-dimensional medium

Let us begin by considering $\mu = \mu(x)$ alone; that is, when the medium varies only in the incident wave direction. From (2.1) we find that the incident wave continues through the medium without change in direction. Its average phase velocity, obtained most readily from (2.1), is

$$\begin{aligned}\overline{\frac{\partial X_x}{\partial t}} &= \overline{c\{X_x(t)\}} = c_0\overline{(1+\mu)^{\frac{1}{2}}} \\ &= c_0[1 + \tfrac{1}{2}\overline{\mu\{X_x(t)\}} - \tfrac{1}{8}\overline{\mu^2\{X_x(t)\}} + \ldots].\end{aligned} \tag{3.1}$$

Even though the medium is to be considered statistically homogeneous, the averaged values of μ that appear in this equation are not zero since they represent averages following a phase point. Rather, $\overline{\mu(x)}$ is zero, by our definition of c_0. This point is developed in the Appendix. It indicates that the mean retardation $\overline{\Delta}_x \neq 0$, and we now proceed to calculate this quantity.

To the first order, from (2.11 *a*),

$$\overline{X_x^{(1)}} = 0. \tag{3.2}$$

For the next order, we examine the one-dimensional form of (2.12 *a*)

$$\frac{\partial^2 X_x^{(2)}}{\partial t^2} = 2c_0 \frac{\partial X_x^{(1)}}{\partial t}\frac{\partial \mu}{\partial x} + \frac{c_0^2}{2} X_x^{(1)} \frac{\partial^2 \mu}{\partial x^2} - \frac{c_0^2}{2}\frac{\partial \mu^2}{\partial x}. \tag{3.3}$$

Recognizing that

$$\left.\frac{\partial \mu}{\partial x}\right|_{x=X^{(0)}(t)} = \frac{1}{c_0}\frac{\partial \mu(c_0 t)}{\partial t}, \tag{3.4}$$

and substituting for $X_x^{(1)}$, we find that the last equation reduces to

$$\frac{\partial^2 X_x^{(2)}}{\partial t^2} = \frac{1}{4}\frac{\partial^2 \mu(c_0 t)}{\partial t^2}\int_0^{c_0 t} \mu(\sigma)\,\partial\sigma. \tag{3.5}$$

Integrating once, we have the relative phase velocity to its first non-zero order,

$$\frac{\partial X_x^{(2)}}{\partial t} = \frac{1}{4}\frac{\partial \mu(c_0 t)}{\partial t}\int_0^{c_0 t} \mu(\sigma)\,d\sigma - \tfrac{1}{8}c_0\,\mu^2(c_0 t). \tag{3.6}$$

If we denote the auto-correlation function of $\mu(x)$ by

$$R(x'') = \overline{\mu(x')\,\mu(x'+x'')}, \tag{3.7}$$

then the mean velocity is

$$\frac{\overline{\partial X_x^{(2)}}}{\partial t} = \tfrac{1}{4}\,c_0\{R(c_0 t) - \tfrac{3}{2}\overline{\mu^2}\}. \tag{3.8}$$

In this one-dimensional medium, the approximation of $\overline{\Delta}_x$ by $\overline{X_x^{(2)}}$ can be shown to be equally good for all times (see the Appendix). The right-hand side of (3.8) is strictly negative, indicating that the mean wave front is constantly being retarded when compared to its propagation through a uniform medium. The phase point lingers in regions where the phase velocity is low, so that its mean velocity is less than the spatial mean velocity in the medium.

4. Wave retardation by a two-dimensional medium

If we now let $\mu = \mu(x, z)$, the ray no longer remains parallel to the incident direction. The two-dimensional variations in local wave speed cause a phase point to meander, bending the rays towards regions where the local phase velocity is least. This increases the path length corresponding to a given distance in the incident direction and further retards the progress of the wave front. It is readily seen, from the symmetrical forms of (2.11 *b*) and (2.12 *b*), that considering a third dimension would only add to computational difficulties rather than present qualitatively new phenomena.

As before, $\overline{X_x^{(1)}} = 0$, and we look to $\overline{X_x^{(2)}}$ for an approximation to $\overline{\Delta}_x$. From a comparison of (2.12*a*) and (3.3) we find the additional relative displacement of the wave front (because of variations in the z-direction) to be given by

$$\frac{\partial^2 X_x^{(2)}}{\partial t^2} = \tfrac{1}{2}c_0^2 X_z^{(1)} \frac{\partial^2 \mu}{\partial x \partial z} + c_0 \frac{\partial X_z^{(1)}}{\partial t} \frac{\partial \mu}{\partial z}. \tag{4.1}$$

Substituting from (2.11*b*) and then integrating, we have

$$\frac{\partial^2 X_x^{(2)}}{\partial t^2} = -\frac{c_0^2}{4} \frac{\partial^2 \mu}{\partial x \partial z}\bigg|_{\mathbf{x}^{(0)}(t)} \int_0^{c_0 t} \int_0^{\rho} \frac{\partial \mu(\sigma, z)}{\partial z}\bigg|_{z=0} d\sigma\, d\rho - \frac{c_0^2}{2} \frac{\partial \mu}{\partial z}\bigg|_{\mathbf{x}^{(0)}(t)} \int_0^{c_0 t} \frac{\partial \mu(\sigma, z)}{\partial z}\bigg|_{z=0} d\sigma, \tag{4.2}$$

$$\frac{\partial X_x^{(2)}}{\partial t} = -\frac{c_0}{4} \int_0^{c_0 t} \int_0^{\rho} \left[\frac{\partial \mu(c_0 t, z)}{\partial z} \frac{\partial \mu(\sigma, z)}{\partial z} + \frac{\partial \mu(\rho, z)}{\partial z} \frac{\partial \mu(\sigma, z)}{\partial z}\right]_{z=0} d\sigma\, d\rho. \tag{4.3}$$

Taking the ensemble average and noting that

$$\overline{\frac{\partial \mu(x', z')}{\partial z'}\bigg|_{z'=0} \frac{\partial \mu(x'', z'')}{\partial z''}\bigg|_{z''=0}} = -\frac{\partial^2 R(x' - x'', z)}{\partial z^2}\bigg|_{z=0}, \tag{4.4}$$

we find that (4.3) becomes

$$\begin{aligned} \overline{\frac{\partial X_x^{(2)}}{\partial t}} &= \frac{c_0}{4} \int_0^{c_0 t} \int_0^{\rho} \frac{\partial^2}{\partial z^2} [R(c_0 t - \sigma, z) + R(\rho - \sigma, z)]|_{z=0}\, d\sigma\, d\rho \\ &= \frac{c_0^2 t}{4} \int_0^{c_0 t} \frac{\partial^2 R(u, z)}{\partial z^2}\bigg|_{z=0} du. \end{aligned} \tag{4.5}$$

$R(x, z)$ is the two-dimensional auto-correlation defined in the manner of (3.7).

The total mean retardation of the phase velocity, for $\mu = \mu(x, z)$, is the combined result of (3.8) and (4.5)

$$\overline{\frac{\partial X_x^{(2)}}{\partial t}} = \tfrac{1}{4} c_0 \{R(c_0 t, 0) - \tfrac{3}{2}\overline{\mu^2}\} + \frac{c_0^2 t}{4} \int_0^{c_0 t} \frac{\partial^2 R(u, z)}{\partial z^2}\bigg|_{z=0} du. \tag{4.6}$$

The total mean retardation of the wave front is

$$\overline{X_x^{(2)}} = \frac{1}{4}\left(\int_0^{c_0 t} R(u, 0)\, du - \tfrac{3}{2}\overline{\mu^2} c_0 t\right) + \frac{c_0 t}{4} \int_0^{c_0 t} (c_0 t - u) \frac{\partial^2 R(u, z)}{\partial z^2}\bigg|_{z=0} du. \tag{4.7}$$

Limiting forms

We now wish to examine the initial, and the long-time retardation. These ranges are defined relative to the appropriate integral length scale of the random medium, L_x, where

$$L_x \equiv (\overline{\mu^2})^{-1} \int_0^{\infty} R(x, 0)\, dx. \tag{4.8}$$

When $c_0 t / L_x$ is sufficiently small compared to unity, the asymptotic form of the retardation is

$$\begin{aligned} \overline{\Delta}_x &\simeq \overline{X_x^{(2)}} \\ &= -\frac{c_0 t}{8}\left(\overline{\mu^2} - (c_0 t)^2 \left[\frac{1}{3} \frac{\partial^2 R(x, 0)}{\partial x^2}\bigg|_{x=0} + \frac{\partial^2 R(0, z)}{\partial z^2}\bigg|_{z=0} + \ldots\right]\right). \end{aligned} \tag{4.9}$$

Since $\{\partial^2 R(x,0)/\partial x^2\}_{x=0}$ and $\{\partial^2 R(0,z)/\partial z^2\}_{z=0}$ are strictly negative, we observe that each term contributes to retarding the progress of the wave. Initially only the first term need be retained and the transverse variations of the medium have a negligible influence.

Of greater interest is the position of the wave front after a relatively long time has elapsed. For this we want the asymptotic expression for $\overline{X_x^{(2)}}$ for large t. Unless it can be shown otherwise, the best we can assume is that $\overline{X_x^{(2)}}$ can be equated to $\overline{\Delta}_x$ for large but bounded times. The bound represents the limit beyond which we cannot ignore further terms of the series (2.13) for $\overline{\Delta}_x$. Since each term in the series contains the scaling parameter μ', we will formally designate this upper bound as $T(\mu')$ and postpone a discussion of its value until § 6.

It appears that a sufficient restriction for obtaining a simple long time form of $\overline{X_x^{(2)}}$, and other quantities we shall later be interested in, is to assume that the already statistically homogeneous medium has a separable correlation function

$$R(x,z) = R(x,0)\,R(0,z). \tag{4.10}$$

Then

$$\left.\frac{\partial^2 R(x,z)}{\partial z^2}\right|_{z=0} \equiv -\frac{1}{\lambda_z^2}R(x,0), \tag{4.11}$$

where λ_z is a length scale associated with the inhomogeneity of the medium in a direction perpendicular to the incident direction. We can expect λ_z to be smaller than the integral length scale, L_z, because of its dependence on the smaller-scale inhomogeneities. In a Gaussian correlation function the two would be equal.

Defining one additional criterion, the time t_0 such that

$$\frac{1}{\overline{\mu^2}}\int_0^{c_0 t} R(x,0)\,dx \simeq L_x \quad \text{for} \quad t > t_0, \tag{4.12}$$

we are now prepared to estimate the wave-front retardation for large, but not unbounded, times.

From (4.6), and limiting ourselves to the class of separable correlation functions, we have

$$\frac{\overline{\partial X_x^{(2)}}}{\partial t} = -\tfrac{3}{8}c_0\overline{\mu^2}\left(1-\frac{2}{3}\frac{R(c_0 t,0)}{\overline{\mu^2}}\right)-\frac{c_0^2 t}{4\lambda_z^2}\overline{\mu^2}L_x \quad \text{for} \quad t > t_0, \tag{4.13}$$

and

$$\overline{\Delta}_x \simeq \int_0^t \frac{\overline{\partial X_x^{(2)}}}{\partial t}\,dt$$

$$\simeq -\frac{\overline{\mu^2}}{8}\left(3+\frac{c_0 t L_x}{\lambda_z^2}\right)c_0 t \quad \text{for} \quad t_0 \ll t \leqslant T(\mu'). \tag{4.14}$$

The two terms contributing to the retardation of the wave front are easily recognizable. The relative importance of phase point meandering off the incident direction is seen to depend equally on the ratios $c_0 t/L_x$ and L_x^2/λ_z^2. In general, it will tend to dominate the result. As $L_x/\lambda_z \to 0$, though, we observe the return to the one-dimensional solution.

Mean square displacement

A second measure of the propagation of the wave in the incident direction is its mean-square displacement about the unperturbed position. This quantity becomes of critical importance if one wishes to observe $\overline{\partial\Delta_x/\partial t}$ and $\overline{\Delta_x}$ on a particular trial of any single member of the ensemble. If the root-mean square dispersion is large compared to the mean retardation of the wave front, then a critical observation on a small number of trials (or over a short part of the wave front) is futile.

From (2.11*a*) we note that a first approximation for $\overline{\Delta_x^2}$ is $\overline{X_x^{(1)2}}$ and we find

$$X_x^{(1)}\frac{\partial X_x^{(1)}}{\partial t} = \left(\frac{1}{2}\int_0^{c_0 t}\mu(\sigma,0)\,d\sigma\right)(\tfrac{1}{2}c_0\mu(c_0 t,0)),$$

or

$$\overline{X_x^{(1)2}} = \frac{1}{2}\int_0^{c_0 t}(c_0 t-\rho)\,R(\rho,0)\,d\rho. \tag{4.15}$$

For the two asymptotic time approximations we examined earlier, we now have

$$\overline{\Delta_x^2} \simeq \overline{X_x^{(1)2}} = \left(\frac{c_0 t}{2}\right)^2\overline{\mu^2} \quad \text{for} \quad \frac{c_0 t}{L_x} \ll 1 \tag{4.16}$$

and

$$\overline{\Delta_x^2} \simeq \tfrac{1}{2}c_0 t L_x\overline{\mu^2} \quad \text{for} \quad t_0 \ll t \leqslant T(\mu'), \tag{4.17}$$

where $T(\mu')$ is the time beyond which further terms in the expansion (2.13) are required.

This expression we have found for mean-square displacement about the unperturbed position is very simple. It is not until we go to further terms in the series expansion for $\overline{\Delta_x^2}$ ($\overline{X_x^{(2)}X_x^{(1)}}, \overline{X_x^{(2)2}}$, etc.) that the phase point meandering off the incident direction and the distinction of the medium's refractive index along the ray to that on the unperturbed path will enter the discussion. The desirable calculation of $\overline{X_x^{(2)2}}$ (the next important term if $\overline{\mu^4}/\mu'^4 \gg \overline{\mu^3}/\mu'^3$, as is likely) is formidable and will not be attempted here.

5. Phase point scattering in a two-dimensional medium

We have already noted that in the two or more dimensional medium, the rays meander from the incident direction. To the first order, we see from (2.11*b*) that

$$\overline{X_z^{(1)}(t)} = 0. \tag{5.1}$$

Going to the next order to find an approximate expression for $\overline{\Delta_z(t)}$ we obtain, from (2.12*b*),

$$\begin{aligned}\frac{\partial^2 X_z^{(2)}}{\partial t^2} &= \frac{c_0^2}{4}\frac{\partial^2\mu(c_0 t,z)}{\partial z^2}\bigg|_{z=0}\int_0^{c_0 t}\int_0^{\rho}\frac{\partial\mu(\sigma,z)}{\partial z}\bigg|_{z=0}d\sigma\,d\rho \\ &\quad -\frac{c_0}{4}\frac{\partial^2\mu(c_0 t,z)}{\partial z\,\partial t}\bigg|_{z=0}\int_0^{c_0 t}\mu(\sigma,0)\,d\sigma-\frac{c_0}{2}\frac{\partial\mu(c_0 t,0)}{\partial t}\int_0^{c_0 t}\frac{\partial\mu(\sigma,z)}{\partial z}\bigg|_{z=0}d\sigma. \end{aligned} \tag{5.2}$$

Integrating once, taking an ensemble average and making relatively obvious changes in variables, the mean dispersion is given by

$$\frac{\partial \overline{X_z^{(2)}}}{\partial t} = -\frac{c_0}{4}\int_0^{c_0 t}\int_0^{\nu}\int_0^{\rho}\frac{\partial^3}{\partial z^3}R(\nu-\sigma, z)\big|_{z=0}\,d\sigma\,d\rho\,d\nu$$
$$-\frac{c_0}{2}\int_0^{c_0 t}\frac{\partial}{\partial z}\left[R(c_0 t-\sigma, -z)+\tfrac{1}{2}R(c_0 t-\sigma, z)\right]\big|_{z=0}\,d\sigma$$
$$=\frac{c_0}{4}\int_0^{c_0 t}\int_0^{w}\int_w^{w-v}\frac{\partial^3 R(u,z)}{\partial z^3}\bigg|_{z=0}du\,dv\,dw+\frac{c_0}{4}\int_0^{c_0 t}\frac{\partial R(u,z)}{\partial z}\bigg|_{z=0}du. \tag{5.3}$$

To obtain the above expression it should be noted that the fact that the correlation function $R(\mathbf{r})$ is an even function of $\mathbf{r}$ for a statistically homogeneous medium implies that $\partial R(x,z)/\partial z|_{z=0}$ is an odd function of x.

If the correlation function is invariant to reflexions about the incident direction ($R(x,z) = R(x,-z)$), then $\overline{X_z^{(2)}} = 0$. Thus, not only homogeneity, but also a statistical symmetry is required for the phase point to be found along the incident direction, on the average. In general, (5.3) will describe a lateral drift of the wave front.

The mean-square dispersion about the mean is obtained from (2.11*b*) and using our previous techniques, is found to be described by

$$\frac{\partial \overline{X_z^{(1)2}}}{\partial t} = \frac{c_0}{8}\int_0^{c_0 t}\int_0^{c_0 t}\int_w^{w-v}\frac{\partial^2 R(u,z)}{\partial z^2}\bigg|_{z=0}du\,dv\,dw. \tag{5.4}$$

Limiting forms

Re-introducing the restriction to a separable correlation function, to make use of (4.11), enables (5.4) to be integrated twice by parts to yield

$$\frac{\partial \overline{X_z^{(1)2}}}{\partial t} = \frac{1}{8}\frac{c_0^2 t}{\lambda_z^2}\int_0^{c_0 t}(c_0 t-w)\,R(w,0)\,dw. \tag{5.5}$$

The short- and long-time asymptotic forms are

$$\overline{\Delta_z^2} \simeq \overline{X_z^{(1)2}} = \frac{1}{64}\frac{c_0^4 t^4}{\lambda_z^2}\overline{\mu^2} \quad \text{for} \quad \frac{c_0 t}{L_x} \ll 1, \tag{5.6}$$

and

$$\overline{\Delta_z^2} \simeq \frac{1}{24}\frac{c_0^3 t^3 L_x}{\lambda_z^2}\overline{\mu^2} \quad \text{for} \quad t_0 \ll t \leqslant T(\mu'). \tag{5.7}$$

6. A sufficient condition for the 'long time' approximations to apply

In the two preceding sections, the exact asymptotic forms of $\overline{\mathbf{X}^{(2)}(t)}$ and $\overline{\mathbf{X}^{(1)2}(t)}$ were obtained for times much longer than that required for the slowly perturbed plane wave to traverse a characteristic length of the medium in the incident direction. Under the formal restriction that the elapsed time was, in turn, smaller than some other time $T(\mu')$, the above statistical quantities were equated to the mean and mean-square dispersion of a phase point about its unperturbed location at each instant of time. A possible estimate of the minimum value of T can now be suggested.

The necessary condition for retaining only the lower-order terms from the basic ray equation is that these terms dominate the result. It should be sufficient, then, that the phase point be restricted in its meandering relative to the length scale in each respective direction. It is not at all obvious that such a stringent criterion is necessary. With this in mind, we estimate a value for T by requiring that

$$\max\{L_x^{-1}(\overline{X_x^{(1)2}})^{\frac{1}{2}},\ L_z^{-1}(\overline{X_z^{(1)2}})^{\frac{1}{2}}\} \leqslant 1 \quad \text{for} \quad t \leqslant T(\mu'). \tag{6.1}$$

Substituting from (4.17) and (5.7),

$$\max\left\{\left(\frac{c_0 t}{2L_x}\right)^{\frac{1}{2}}\mu',\ \frac{1}{\sqrt{24}},\ \left(\frac{L_x}{L_z}\right)\left(\frac{L_x}{\lambda z}\right)\left(\frac{c_0 t}{L_x}\right)^{\frac{3}{2}}\mu'\right\} \leqslant 1. \tag{6.2}$$

An exception to this would be when discussing $\overline{\Delta_x(t)}$ as given by (4.14). There it is sufficient for only $(\overline{\Delta_z^{(1)2}})^{\frac{1}{2}}/L_z$ to be small.

7. An application: shallow water waves

The above analysis has almost immediate application in the study of certain types of shallow-water waves, including the tsunami, and a brief discussion should prove interesting to those familiar with the subject. Rather than simply refer

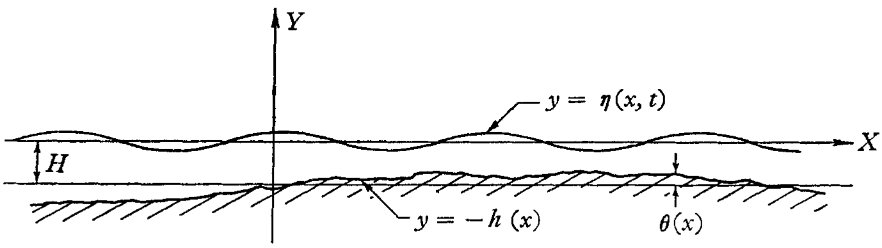

FIGURE 1. Shallow-water wave over a random bottom.

to the work of Lowell (1949) and Keller (1958), who explicitly derived the eikonal equation for shallow-water waves over variable depths, it will better serve our purpose to rederive the equation for the surface elevation $y = \eta(x, z, t)$

$$\frac{\partial^2\eta}{\partial t^2} - \nabla.(gh\nabla\eta) = 0, \tag{7.1}$$

where $y = -h(x, z)$ is the randomly variable local depth and g is the local gravitational constant. Our object will be to obtain the statistical conditions implied by the use of (7.1) and, furthermore, the additional restrictions needed to reduce it to the form of (1.1). Of particular importance is the relative magnitude of the non-linear wave interaction and the wave perturbation by the random bottom. It shall be sufficient to derive the one-dimensional form of (7.1), by limiting ourselves to systems in which the integral length scale of h in the incident direction is at least as small as in any other direction. The result can then be generalized.

Consider an incident surface wave of wave-number k and phase velocity c_0, entering a region where the depth is a statistically homogeneous function of position with a root-mean-square variation θ' about a mean depth H, and a characteristic length scale L. By assuming only that the water is shallow

($kH \ll 1$), the surface elevation of an inviscid fluid in a gravitational field is described by the equation of motion in the x-direction,

$$u_{,t} + uu_{,x} = -g\eta_{,x} \tag{7.2}$$

and the continuity equation,

$$[u(\eta + h)]_{,x} = \eta_{,t}, \tag{7.3}$$

where $u = u(x,t)$ is the fluid particle velocity in the x-direction (see Stoker 1957, § 2). The two equations combine to yield, without further approximations,

$$\eta_{,tt} - gH\eta_{,xx} = g\eta_{,xx}(\eta + \theta) + g\eta_{,x}(\eta + \theta)_{,x} - 2(u\eta_{,t})_{,x} - (u^2(\eta + \theta)_{,x})_{,x}. \tag{7.4}$$

By defining an average wave amplitude, η', we can evaluate the relative importance of each term in the above equation.

As the only meaningful time scale in the problem is the incident wave period $(2\pi/kc_0)$, the fluid particle velocity in the y-direction, v, can be shown to be of the magnitude of $kc_0\eta'$. Since the velocity field is irrotational,

$$u = \int v_{,y}\,dx \sim \frac{c_0\eta'}{H}$$

when $kL \gg 1$. With this estimate, and ignoring terms whose magnitudes are smaller than $(\eta'/H)^2$, the relative magnitude of the remaining terms of (7.4) are

$$\left.\begin{array}{ccccccccc} \eta_{,tt} - & c_0^2\eta_{,xx} = & g\eta_{,xx}\eta + & g\eta_{,xx}\theta + & g\eta_{,x}\eta_{,x} + & g\eta_{,x}\theta_{,x} - & 2u_{,x}\eta_{,t} - & 2u\eta_{,tx}. \\ 1 & 1 & \dfrac{\eta'}{H} & \dfrac{\theta'}{H} & \dfrac{\eta'}{H} & \dfrac{\theta'}{H}kL & \dfrac{\eta'}{H} & \dfrac{\eta'}{H} \end{array}\right\} \tag{7.5}$$

Thus, the criterion for ignoring all the non-linear terms is $\eta'/\theta' \ll 1$. If, in addition, the length scale of the bottom is long compared to the incident wavelength ($kL \gg 1$), then we are left with only

$$\eta_{,tt} - c_0^2\eta_{,xx} = g\eta_{,xx}\theta, \tag{7.6}$$

or

$$\eta_{,tt} - c^2\eta_{,xx} = 0,$$

where $c = (gh(x))^{\frac{1}{2}}$ is the local phase speed. Generalizing to the two-dimensional representation, one obtains the inhomogeneous wave equation of (1.1). We now add the restriction that the depth variation from its mean, as measured by θ'/H, is small compared to unity and we have a total of four conditions for applying ray theory to shallow-water waves. (1) $kH \ll 1$, the shallow-water approximation. (2) $\eta'/\theta' \ll 1$, suppressing the non-linear features of the total problem while leaving the randomness of the medium as the perturbing mechanism. (3) $kL \gg 1$ and (4) $\theta'/H \ll 1$, which combine to minimize the changes which occur to a wave in propagating one (surface) wavelength and thereby permit the use of a local plane wave concept.

To indicate that a tsunami travelling over deep ocean may satisfy the above conditions, Katz (1963) calculated two one-dimensional auto-correlations for the depth variation in the Pacific Ocean. Statistical homogeneity was assumed. On the basis of the data, one may speculate that $L = 500$ mi. and $\theta'/H = 10^{-1}$ are representative values. A typical tsunami may be 1 ft. high and 100 miles long (after Cagle 1962). If we take the mean depth as 1·5 mi., then we find that $kH = O[10^{-1}]$, $kL = O[10]$ and $\eta'/\theta' = O[10^{-3}]$. The magnitudes of the parameters involved, while not totally convincing, definitely suggest that the ray theory may be applicable.

If we now attempt to calculate the retardation and dispersion of a tsunami across a 6000 mile length of ocean (roughly approximating a great circle arc to a line on a plane) one finds, from (4.14), that the wave front is retarded about a quarter of a wave period due to the variable depth along the incident direction, or several minutes. Including transverse variations in depth, and presuming the truncated expansion to be valid to this point, the retardation is then increased by an order of magnitude of $(L_x/\lambda_z)^2$ wave periods. At the present time, there is insufficient oceanographical data to attempt an estimate of this ratio. Again assuming the time bound has not been overstepped (this depends on knowing the magnitude of both λ_z/L_x and L_z/L_x, as seen in (6.2)), the root-mean-square deviation of the wave front about its unperturbed position is of the order of a wave period while the root-mean-square dispersion of a phase point from the incident direction is $4L_x/\lambda_z$ wave periods. The question of whether the analysis can be used for predictions over as long a distance as attempted here will have to await further systematic observations of the topography of the ocean bed.

This research was supported by the Mechanics Branch, U.S. Office of Naval Research, under Contract Nonr 248(56), and was assisted by a personal fellowship grant from the Bendix Aviation Corporation.

I am indebted to Dr S. Corrsin for his critical assistance throughout this work, beginning with his initial suggestion of marking the phase point in order to trace the propagation of a shallow-water wave. In addition, Dr O. M. Phillips offered helpful suggestions while the work was in progress.

Appendix

In this appendix, an alternative method will be given for the wave retardation in a one-dimensional medium. When $\mu = \mu(x)$, (4.14) reduces to give the mean wave front retardation as

$$\overline{\Delta_x} \simeq -\tfrac{3}{8}\overline{\mu^2}\, c_0 t \quad \text{for} \quad t_0 \ll t \leqslant T(\mu').$$

The average phase velocity can be expressed as

$$\overline{\frac{\partial X_x}{\partial t}} = c_0(1 - \tfrac{3}{8}\overline{\mu^2} + \ldots) \quad \text{for} \quad t_0 < t \leqslant T(\mu'). \tag{A 1}$$

We now propose to demonstrate that this approximation is equally good for *all* times sufficiently larger than t_0.

For this simple medium we have noted that the average phase velocity can be written almost immediately as

$$\overline{\frac{\partial X_x}{\partial t}} = \overline{c\{X_x(t)\}} = c_0[1 + \tfrac{1}{2}\overline{\mu\{X_x(t)\}} - \tfrac{1}{8}\overline{\mu^2\{X_x(t)\}} + \ldots].$$

Putting aside our previous analysis, we examine what is meant by the pseudo-Lagrangian ensemble averages of the moments of μ ($\overline{\mu\{X_x(t)\}}, \overline{\mu^2\{X_x(t)\}}$, etc.).

Assuming we are permitted to replace ensemble averages by space averages,

$$\begin{aligned}\overline{\mu(x)} &= \lim_{s\to\infty}\frac{1}{s}\int_0^s \mu(x)\,dx \\ &= \lim_{s\to\infty}\frac{c_0}{s}\int_0^{s/c_0}\mu\{X_x^{(0)}(t)\}\,dt \\ &= \overline{\mu\{X_x^{(0)}(t)\}}. \end{aligned} \tag{A 2}$$

In a sense, $\overline{\mu\{X_x^{(0)}\}}$ is an unbiased average of $\mu(x)$, for sufficiently long sampling time. In this manner, $\overline{\mu\{X_x(t)\}}$ can be thought of as a biased sampling of $\mu(x)$. The bias arises from the tagged phased point favouring regions where $\mu(x)$ is negative, where it travels more slowly, as contrasted with the positive regions (equally probable in x) through which it 'speeds'. Therefore we might anticipate $\overline{\mu\{X_x(t)\}}$, for instance, to be monotonically decreasing with time (monotonic, because longer time permits greater biasing). When the wave has traversed several characteristic lengths of the medium, an asymptotic value is approached.

$$\overline{\mu\{X_x(t)\}} = \frac{\int_0^{c_0 t}\mu(x)\,[c(x)]^{-1}\,dx}{\int_0^{c_0 t}[c(x)]^{-1}\,dx}, \tag{A 3}$$

where $c(x)^{-1}$ is the weight given $\mu(x)$ for the increment of path length dx at x. Replacing $c(x)$ by $c_0(1+\mu(x))^{\frac{1}{2}}$, and expanding $(1+\mu(x))^{-\frac{1}{2}}$ for $|\mu(x)| \ll 1$, we find

$$\begin{aligned}\overline{\mu\{X_x(t)\}} &= \int_0^{c_0 t}\mu(x)\,(1-\tfrac{1}{2}\mu(x)+\ldots)\,dx\Big/\int_0^{c_0 t}(1-\tfrac{1}{2}\mu(x)+\ldots)\,dx \\ &\to -\tfrac{1}{2}\overline{\mu^2}+\ldots \quad \text{as} \quad t\to\infty, \end{aligned} \tag{A 4}$$

where the additional terms are, in many physical instances, of magnitude $(\overline{\mu^2})^2$. To evaluate the asymptotic values of the higher moments of $\mu\{X_x(t)\}$ we extend the biased sampling average

$$\overline{\mu^n\{X_x(t)\}} = \frac{\int_0^{c_0 t}\mu^n(x)\,[c(x)]^{-1}\,dx}{\int_0^{c_0 t}[c(x)]^{-1}\,dx}, \tag{A 5}$$

and one calculates that

$$\overline{\mu^2\{X_x(t)\}} \to \overline{\mu^2}+\ldots \quad \text{as} \quad t\to\infty. \tag{A 6}$$

For $\mu' \ll 1$, this is as much as we need and, after substitution, (3.1) becomes

$$\frac{\partial \overline{X_x}}{\partial t} \simeq c_0(1-\tfrac{3}{8}\overline{\mu^2}) \quad \text{for large } t,$$

which is (A 1), our previous result. Because the right-hand side will never be a function of time no matter how many terms one calculates, there is no need for the upper bound on t in this particular case. Thus, for $t \gg t_0$ and $\mu' \ll 1$, (4.14) is a good approximation as long as the incident wave has not been appreciably scattered transversely.

REFERENCES

Batchelor, G. K. 1957 *Symposium in Naval Hydrodynamics.* Nat. Res. Counc. (Wash.), Publ. no. 515, 409.

Cagle, B. J. 1962 *Naval Res. Rev.* (Feb.), 15.

Chernov, L. A. 1960 *Wave Propagation in a Random Medium.* New York: McGraw Hill.

Ellison, T. H. 1951 *J. Atmos. Terres. Phys.* **2**, 14.

Katz, E. J. 1962 *J. Geophys. Res.* **67**, 4713.

Katz, E. J. 1963 *Deep Sea Res.* (in the Press).

Keller, J. B. 1958 *J. Fluid Mech.* **4**, 607.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics.* London: Pergamon Press.

Lowell, S. C. 1949 *Comm. Pure Appl. Math.* **2**, 275.

Muchmore, R. B. & Wheelon, A. D. 1955 *Proc. Inst. Rad. Engrs,* **43**, 1437.

Stoker, J. J. 1957 *Water Waves.* New York: Interscience.

[*Note added in proof:* A study by J. B. Keller (*Proc. Symp. Appl. Math.* **13** (1962), Amer. Math. Soc.) proposes a ray theory which resembles the one discussed here. The principal difference is that in Keller's work the ray is described as a function of arc-length along the ray, whereas in the present work it is a function of time. The former formulation omits the effect of variable wave speed along the ray path, which we have shown to diminish the arc-length travelled relative to that travelled in a uniform medium.]